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LETTER TO THE EDITOR

Quantum tunnelling effect for the inverted Caldirola–Kanai Hamiltonian

S Baskoutas and A Jannussis†

Department of Physics, Patras University, 26110 Patras, Greece

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Abstract. In order to study the behaviour of the inverted Caldirola–Kanai Hamiltonian in the quantum tunnelling effect, we consider a wavepacket as the initial state and we calculate exactly the probability density. We also obtain the transmission and reflection probability, the expectation value of the particle's energy and the sojourn time which appears to be an increasing function of the dissipation parameter γ. A similar result is also obtained by considering a normalized plane wave as the initial state.

The problem of the time-dependent oscillator has long been a research area of considerable interest because of its various applications in different areas of physics. For instance, in molecular physics, quantum chemistry and quantum optics many quantum mechanical effects are treated phenomenologically by means of the time-dependent parameters in the Hamiltonian of the damped harmonic oscillator [1, 2], i.e. the well known Caldirola–Kanai Hamiltonian

$$H=\frac{p^2}{2m}\,\mathrm{e}^{-\gamma t}+\frac{m}{2}\,\omega^2q^2\,\mathrm{e}^{\gamma t}. \tag{1}$$

Recently, extensive efforts have been made to obtain exact solutions to the Schrödinger equation for oscillator systems with time-dependent Hamiltonians [3, 4]. In this letter we attempt to present the behaviour in the quantum tunnelling effect of a Hamiltonian which is called the inverted Caldirola–Kanai Hamiltonian [5] and which is formally obtainable from (1) by the replacement

$$\omega\to\mathrm{i}\omega. \tag{2}$$

Thus this Hamiltonian has the following form:

$$H=\frac{p^2}{2m}\,\mathrm{e}^{-\gamma t}-\frac{m}{2}\,\omega^2q^2\,\mathrm{e}^{\gamma t}. \tag{3}$$

Of course in spite of many useful analogies between (1) and (3), the physics of the two cases is very different; the energy eigenstates of (3) are not square integrable and they are doubly degenerate, e.g. with respect to incidence from left or right or alternatively with respect to parity [5]. When γ tends to zero we obtain the Hamiltonian of the inverted harmonic oscillator. Actually the inverted harmonic oscillator besides its applications in masers [6], has also been used in reactive scattering [7], due to the

† Also at: IBR, PO Box 1577, Palm Harbor, Florida 34682-1577, USA.

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fact that for this kind of potential the tunnelling time does not diverge as in the case of a square barrier at threshold in the semiclassical limit analysis [8, 9]. Also we must notice that, similarly to the Hamiltonian of the inverted harmonic oscillator, the inverted Caldirola–Kanai Hamiltonian does also produce squeezed states as has been proved by Jannussis *et al* [5, 10].

In order to study the quantum tunnelling effect for the Hamiltonian (3) we consider in this letter a particle in the field of our potential barrier in a state initially prepared in the form of a wavepacket

$$\Psi_{(q_0,p_0)}(q)=(2\pi\sigma^2)^{-1/4}\exp\left(-\frac{1}{4\sigma^2}(q-q_0)^2+\frac{\mathrm{i}}{\hbar}p_0q\right). \tag{4}$$

Such a state establishes a certain initial probability distribution of finding the particle in a region around the point q_0, which throughout this work will be assumed to be on the left-hand side of the barrier. Thus the most significant part of the probability distribution lies mainly on the left-hand side of the barrier and furthermore the distribution may extend through a tail on the other side. Our approach to the tunnelling problem will rely on processes leading to transport of the above initial probability from the left-hand side to the right-hand side of the barrier.

Provided the exact propagator is obtained, the wavefunction of our system can be calculated, according to the formula

$$\Psi_{(q_0,p_0)}(q,t)=\int_{-\infty}^{+\infty}G(q,t/q',0)\Psi_{(q_0,p_0)}(q')\,\mathrm{d}q'. \tag{5}$$

As is well known, the path integral formalism provides an approach to the calculation of the propagator associated with the one-dimensional classical quadratic action of an oscillator system with time-dependent Hamiltonian [11]. A different approach to the problem, which is based on Lie-algebraic methods has been developed in [12], exploiting the underlying SU(1, 1) structure of the quadratic Hamiltonians

$$H=Z(t)\frac{p^2}{2m}+\frac{Y(t)}{2}(qp+pq)+\frac{m}{2}\omega^2X(t)q^2. \tag{6}$$

In the present letter, adopting the formalism of [12, 13], we obtain the propagator of the SU(1, 1) Hamiltonian (3), which has the following form:

$$\begin{aligned}G(q,t/q',0)=\mathrm{e}^{\gamma t/4}&\left(\frac{m\Omega}{2\pi\mathrm{i}\hbar\sinh\Omega t}\right)^{1/2}\exp\left(\frac{\mathrm{i}m\omega}{2\hbar\cosh(\Omega t+\varphi)\sinh\Omega t}\right.\\&\times[\mathrm{e}^{\gamma t}(\cosh^2\varphi+\sinh^2\Omega t)q^2+q'^2\cosh^2(\Omega t+\varphi)\\&\left.-2\,\mathrm{e}^{\gamma t/2}\cosh\varphi\cosh(\Omega t+\varphi)qq']\right)\end{aligned} \tag{7}$$

with

$$\Omega=(\omega^2+\gamma^2/4)^{1/2}\qquad\text{and}\qquad\varphi=\tanh^{-1}\frac{\gamma}{2\Omega}. \tag{8}$$

As can be easily seen, for $\gamma\to0$ we obtain the propagator of the inverted harmonic oscillator [14]. Also it must be noticed that the above propagator (7) is not diagonal in q and q' and does not satisfy the time reversal property

$$G(q,t/q',0)=G^*(q',0/q,t). \tag{9}$$

The breaking of the reciprocity condition (9) is to be expected for dissipative systems of the type (1) or (3), according to [11]. For more details about the dynamical behaviour of systems driven by Hamiltonians (6) the reader is directed to [12].

Thus now we find the solution of the time-dependent Schrödinger equation according to equation (5)

$$\Psi_{(q_0,p_0)}(q,t)=\frac{A(q,t)\sqrt{\pi}\exp\left(-\frac{1}{4\sigma^2}q_0^2\right)}{\left(\frac{1}{4\sigma^2}-\mathrm{i}\frac{m\omega}{2\hbar}\frac{\cosh(\Omega t+\varphi)}{\sinh\Omega t}\right)^{1/2}}\times\exp\left(\frac{1}{4}\frac{\left[\frac{q_0}{2\sigma^2}+\mathrm{i}\left(K_0-\mathrm{e}^{\gamma t/2}\frac{m\omega\cosh\varphi}{\hbar\sinh\Omega t}q\right)\right]^2}{\left[\frac{1}{4\sigma^2}-\mathrm{i}\frac{m\omega\cosh(\Omega t+\varphi)}{2\hbar\sinh\Omega t}\right]}\right) \tag{10}$$

where

$$A(q,t)=\mathrm{e}^{\gamma t/4}\left(\frac{m\Omega}{2\pi\hbar\mathrm{i}\sinh\Omega t}\right)^{1/2}\left(\frac{1}{2\pi\sigma^2}\right)^{1/4}\exp\left(\frac{im\omega}{2\hbar\cosh(\Omega t+\varphi)\sinh\Omega t}q^2\right)$$
$$K_0=\frac{p_0}{\hbar}. \tag{11}$$

The same results can also be obtained, the evolution operator $\hat{U}(t)$ of [12, formulae (15) or (17)] acting directly on the initial state (4).

Then a straightforward calculation yields the probability density of finding the particle in the vicinity of the observation point q at time t,

$$\rho(q,t)=|\Psi(q,t)|^2=(2\pi\sigma^2\Gamma^2(t))^{-1/2}\exp\left(-\frac{(q-q_{\max})^2}{2\sigma^2\Gamma^2(t)}\right) \tag{12}$$

where

$$\Gamma^2(t)=\mathrm{e}^{-2b(t)}\left(1+\frac{\lambda^4\sinh^2\Omega t}{\sigma^4\cosh^2(\Omega t+\varphi)}\right) \tag{13}$$

$$b(t)=\frac{\gamma t}{2}-\ln\frac{\cosh(\Omega t+\varphi)}{\cosh\varphi} \tag{14}$$

and λ is a characteristic length of the scattering process, given by

$$\lambda=\left(\frac{\hbar}{2m\omega}\right)^{1/2} \tag{15}$$

and

$$q_{\max}=\mathrm{e}^{-b(t)}\left(q_0+\frac{p_0}{m\omega}\frac{\sinh\Omega t}{\cosh(\Omega t+\varphi)}\right). \tag{16}$$

Also for $\gamma\to 0$ we obtain exactly the results of [15, 16].

The essential feature in the tunnelling effect lies in that part of probability of finding the particle, initially on the entry side of the barrier, moving into the observation side.

The ratio of the net amount of probability on the right-hand side of the barrier that has migrated into this region in time t over the initial probability of finding the particle on the entry side, is given by the relation [16]

$$T(t)=\left[\int_{-\infty}^{0}\rho(q,0)\,\mathrm{d}q\right]^{-1}\int_{0}^{\infty}[\rho(q,t)-\rho(q,0)]\,\mathrm{d}q \tag{17}$$

where the point zero is the coordinate of the barrier top.

Therefore we obtain the following form of the transmission probability:

$$T(t)=\frac{\mathrm{Erf}(q_{\max}/\sqrt{2}\,\sigma\Gamma(t))-\mathrm{Erf}(q_0/\sqrt{2}\,\sigma)}{1-\mathrm{Erf}(q_0/\sqrt{2}\,\sigma)} \tag{18}$$

where the symbol $\mathrm{Erf}(x)$ represents the error function

$$\mathrm{Erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x \mathrm{e}^{-t^2}\,\mathrm{d}t$$

and according to the formula $R=1-T$ we find the reflection probability

$$R(t)=\frac{\mathrm{Erf}_c(q_{\max}/\sqrt{2}\,\sigma\Gamma(t))}{1-\mathrm{Erf}(q_0/\sqrt{2}\,\sigma)} \tag{19}$$

where $\mathrm{Erf}_c(x)=1-\mathrm{Erf}(x)$.

What actually determines the tunnelling effect is the energy associated with the initial wavepacket, which has to be smaller than the barrier height. Since our system is not conservative, the expectation value of the particle's energy will not remain constant in the course of time. In the present case the expected energy is given by

$$\langle H\rangle=\frac{1}{2m}\left(p_0^2+\frac{1}{4}\frac{\hbar^2}{\sigma^2}\right)\mathrm{e}^{-\gamma t}-\frac{m}{2}\,\omega^2\,\mathrm{e}^{\gamma t}(q_0^2+\sigma^2) \tag{20}$$

or

$$\langle H\rangle=\frac{p_0^2}{2m}\,\mathrm{e}^{-\gamma t}-\frac{m}{2}\,\omega^2 q_0^2\mathrm{e}^{\gamma t}+\tfrac{1}{2}m\omega^2\sigma^2\,\mathrm{e}^{-\gamma t}\left(\frac{\lambda^4}{\sigma^4}-\mathrm{e}^{2\gamma t}\right). \tag{21}$$

In order that the current produced at the observation point be of tunnelling origin we must have $\langle H\rangle<0$, since the top of the barrier for the potential energy (3) is placed at zero as we have already assumed. Thus we must take

$$\frac{1}{2m}\left(p_0^2+\frac{1}{4}\frac{\hbar^2}{\sigma^2}\right)<\frac{m}{2}\,\omega^2\,\mathrm{e}^{2\gamma t}(q_0^2+\sigma^2). \tag{22}$$

As we can easily see the above relation is fully satisfied for large and positive values of the factor γt.

In the following we will calculate the sojourn time or the dwell time as usually referred to in the literature on tunnelling times. Given the many contradictions which exist regarding the correct definition of the tunnelling time [8, 17], we shall not consider this problem here and we will present the tunnelling time as given by the authors of [18] (cf also the appendix of [8]).

Therefore we define as mean total sojourn time the quantity

$$\tau((a,b),-\infty,+\infty;\Psi)=\int_{-\infty}^{+\infty}\mathrm{d}t\int_a^b \mathrm{d}q|\Psi(q,t)|^2 \tag{23}$$

where (a, b) is an interval containing the barrier. Thus with the substitution of equation (12) into (23) we obtain

$$\tau((a,b),-\infty,+\infty;\Psi)=\frac{1}{2}\int_{-\infty}^{+\infty}\mathrm{d}t\left[\mathrm{Erf}\left(\frac{b-q_{\max}}{\sqrt{2}\,\sigma\Gamma(t)}\right)-\mathrm{Erf}\left(\frac{a-q_{\max}}{\sqrt{2}\,\sigma\Gamma(t)}\right)\right]. \tag{24}$$

To a first approximation we may have

$$\mathrm{Erf}\left(\frac{b-q_{\max}}{\sqrt{2}\,\sigma\Gamma(t)}\right)\simeq\frac{b-q_{\max}}{\sqrt{2}\,\sigma\Gamma(t)} \qquad \text{when } b-q_{\max}\ll\sigma \tag{25a}$$

and

$$\mathrm{Erf}\left(\frac{a-q_{\max}}{\sqrt{2}\,\sigma\Gamma(t)}\right)\simeq\frac{a-q_{\max}}{\sqrt{2}\,\sigma\Gamma(t)} \qquad \text{when } a-q_{\max}\ll\sigma \tag{25b}$$

and therefore we take

$$\begin{aligned}\tau((a,b),-\infty,+\infty;\Psi)&=\frac{1}{2}\int_{-\infty}^{+\infty}\mathrm{d}t\,\frac{(b-a)}{\sqrt{2}\,\sigma\Gamma(t)}\\&=\frac{1}{2}\,\frac{(b-a)}{\sqrt{2}\sigma}\int_{-\infty}^{+\infty}\mathrm{e}^{b(t)}\left(1+\frac{\lambda^4}{\sigma^4}\,\frac{\sinh^2\Omega t}{\cosh^2(\Omega t+\varphi)}\right)^{-1/2}\Bigg]\,\mathrm{d}t\end{aligned} \tag{26}$$

where $b(t)$ is defined in equation (14) and λ in equation (15).

Assuming that $\sigma\gg\lambda$ (e.g. an extended wavepacket) we can obtain

$$\begin{aligned}\tau((a,b),-\infty,+\infty,\Psi)&=\frac{1}{2}\,\frac{(b-a)}{\sqrt{2}\sigma}\cosh\varphi\int_{-\infty}^{+\infty}\frac{\mathrm{e}^{\gamma t/2}}{\cosh(\Omega t+\varphi)}\,\mathrm{d}t\\&=\frac{\pi}{2\omega}\,\frac{(b-a)}{\sqrt{2}\sigma}f(\gamma,\omega)\end{aligned} \tag{27}$$

where

$$f(\gamma,\omega)=\frac{1}{\cos(\gamma\pi/4\Omega)}\exp\left(-\frac{\gamma}{2\Omega}\tanh^{-1}\frac{\gamma}{2\Omega}\right). \tag{28}$$

A similar result also appears in the case where we take, as an initial wavefunction, the normalized plane wave, e.g.

$$\Psi(q,0)=\frac{1}{\sqrt{2L}}\mathrm{e}^{\mathrm{i}kq} \qquad \text{(box normalization)}. \tag{29}$$

Thus, according to (5) we find

$$\begin{aligned}\Psi(q,t)=\frac{1}{\sqrt{2L}}\exp\left(\frac{b(t)}{2}\right)\exp\left(\mathrm{i}\,\frac{m\omega}{2\hbar}\,\frac{\sinh\Omega t}{\cosh(\Omega t+\varphi)}\,\mathrm{e}^{\gamma t}q^2\right)\exp(\mathrm{i}k\,\mathrm{e}^{b(t)}q)\\\times\exp\left(-\frac{\mathrm{i}\hbar}{2m\omega}\,\frac{\sinh\Omega t}{\cosh(\Omega t+\varphi)}\,k^2\right)\end{aligned} \tag{30}$$

and

$$|\Psi(q,t)|^2=\frac{1}{2L}\mathrm{e}^{b(t)}. \tag{31}$$

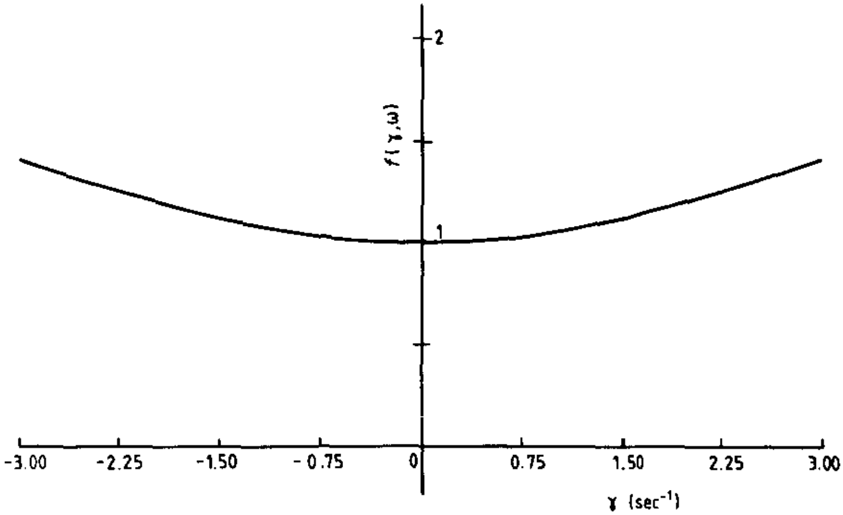

Figure 1. A plot of the function $f(\gamma, \omega)$, for $\omega = 1$ Hz.

Therefore according to (23) we obtain

$$\tau((a, b), -\infty, +\infty; \Psi) = \frac{\pi}{2\omega} \frac{(b-a)}{L} f(\gamma, \omega) \tag{32}$$

and for $L = \sqrt{2}\sigma$ we obtain, exactly, the result (27).

It can be easily found that the existence of the factor $f(\gamma, \omega)$ in the expressions (27) and (32) enhances the sojourn time, taking its minimum value $f(\gamma, \omega) = 1$ for $\gamma = 0$. This situation is more clearly depicted in figure 1. The decrease in the tunnelling process due to the dissipation has also been predicted in a different way by other authors (see, for instance, [9]).

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